FP1 S15

1. $$f(x) = 9x^3 - 33x^2 - 55x - 25$$

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Given that $x = 5$ is a solution of the equation $f(x) = 0$, use an algebraic method to solve $f(x) = 0$ completely.

(5)

$(x-5)$ is a factor

$f(x) = (x-5)(9x^2 + Ax + 5)$

$\Rightarrow -45x^2 + Ax^2 \equiv -33x^2 \quad \therefore A = 12$

$f(x) = (x-5)(9x^2+12x+5)$

$\Rightarrow f(x) = 0 \Rightarrow x = 5 \qquad 9x^2+12x+5=0$

$x^2 + \frac{4}{3}x + \frac{5}{9} = 0$

$\Rightarrow (x+\frac{2}{3})^2 = \frac{4}{9} - \frac{5}{9} = -\frac{1}{9}$

$\therefore x + \frac{2}{3} = \pm\sqrt{\frac{-1}{9}} = \pm\frac{1}{3}i \qquad \therefore x = \frac{-2}{3} \pm \frac{1}{3}i$

If $f(x) = 0 \quad x = 5, -\frac{2}{3}+\frac{1}{3}i, -\frac{2}{3}-\frac{1}{3}i$

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2. In the interval $13 < x < 14$, the equation

$$3 + x\sin\left(\frac{x}{4}\right) = 0, \text{ where } x \text{ is measured in radians,}$$

has exactly one root, α.

(a) Starting with the interval [13, 14], use interval bisection twice to find an interval of width 0.25 which contains α.

(3)

(b) Use linear interpolation once on the interval [13, 14] to find an approximate value for α. Give your answer to 3 decimal places.

(4)

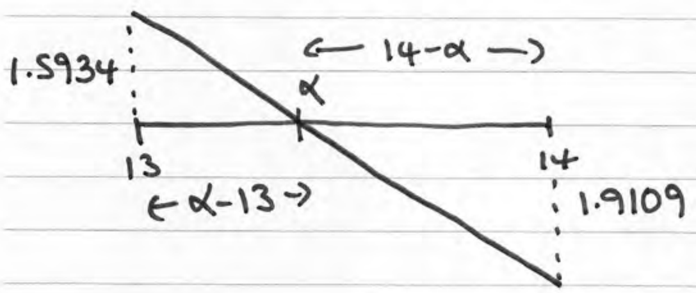
$f(x) = 3 + x\sin(\frac{1}{4}x)$

$x =$	13	13.25	13.5	14
$f(x) =$	1.59	0.75	-0.12	-1.91
	>0	>0	<0	<0

$\underbrace{13.25 \quad 13.5}_{\text{root } \alpha}$

$\therefore \alpha \in (13.25, 13.50)$

b)



$$\frac{14-\alpha}{1.9109} = \frac{\alpha - 13}{1.5934}$$

$22.3076 - 1.5934\alpha$
$=$
$1.9109\alpha - 24.8417$

$\therefore 3.5043\alpha = 47.1493 \quad \therefore \alpha = \underline{13.455}$

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3. (a) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=1}^{n}(r+1)(r+4) = \frac{n}{3}(n+4)(n+5)$$

for all positive integers n.

(5)

(b) Hence show that

$$\sum_{r=n+1}^{2n}(r+1)(r+4) = \frac{n}{3}(n+1)(an+b)$$

where a and b are integers to be found.

(3)

a) $\sum(r+1)(r+4) = \sum r^2 + 5\sum r + 4\sum 1$

$= \frac{1}{6}n(2n^2+3n+1) + 5 \times \frac{1}{2}n(n+1) + 4n$

$= \frac{1}{6}n(2n^2+3n+1) + \frac{15}{6}n(n+1) + \frac{24}{6}n$

$= \frac{1}{6}n[2n^2+3n+1+15n+15+24]$

$= \frac{1}{6}n[2n^2+18n+40]$

$= \frac{1}{3}n(n^2+9n+20) = \frac{1}{3}n(n+4)(n+5)$

b) $\sum_{r=n+1}^{2n}(r+1)(r+4) = \frac{1}{3}(2n)(2n+4)(2n+5) - \frac{1}{3}n(n+4)(n+5)$

$= \frac{1}{3}n[(4n+8)(2n+5) - (n+4)(n+5)]$

$= \frac{1}{3}n[8n^2+36n+40-n^2-9n-20]$

$= \frac{1}{3}n[7n^2+27n+20]$

$= \frac{1}{3}n(7n+20)(n+1)$

4.

$$z_1 = 3\mathrm{i} \text{ and } z_2 = \frac{6}{1+\mathrm{i}\sqrt{3}}$$

(a) Express z_2 in the form $a + \mathrm{i}b$, where a and b are real numbers.

(2)

(b) Find the modulus and the argument of z_2, giving the argument in radians in terms of π.

(4)

(c) Show the three points representing z_1, z_2 and $(z_1 + z_2)$ respectively, on a single Argand diagram.

(2)

a) $Z_2 = \dfrac{6(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \dfrac{6-6i\sqrt{3}}{1+3} = \dfrac{3}{2} - \dfrac{3}{2}i\sqrt{3}$

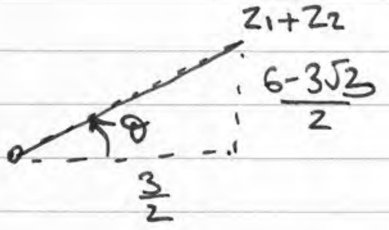
b) $|Z_2| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{27}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$

$* \arg(z_2) = \frac{-\pi}{3}$

c) $Z_1 + Z_2 = 3i + \dfrac{6}{1+i\sqrt{3}} = 3i + \frac{3}{2} - \frac{3}{2}\sqrt{3}\,i$

$= \frac{3}{2} + \left(\frac{6-3\sqrt{3}}{2}\right)i$

$\rightsquigarrow +0.4$

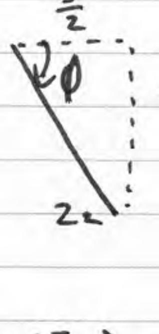


$\tan\theta = \dfrac{\frac{6-3\sqrt{3}}{2}}{\frac{3}{2}}$ $\quad \theta = \tan^{-1}(2-\sqrt{3})$

$\theta = \frac{\pi}{12}$

$Z_2 = \frac{3}{2} - \frac{3}{2}\sqrt{3}\,i$

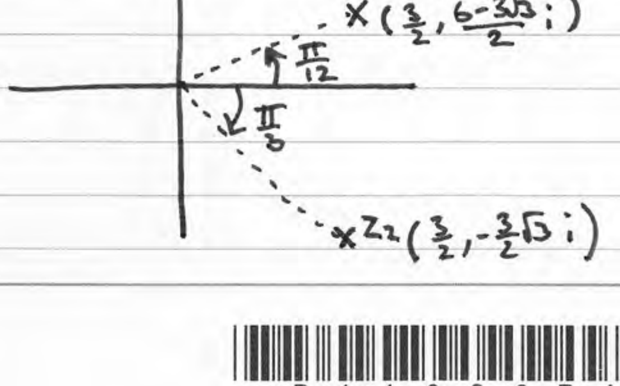
$\hookrightarrow -2.6$



$\tan\phi = \dfrac{\frac{3\sqrt{3}}{2}}{\frac{3}{2}}$

$\Rightarrow \phi = \tan^{-1}(\sqrt{3})$

$\therefore \phi = \frac{\pi}{3}$ $*$



Note
angles not required

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5. The rectangular hyperbola H has equation $xy = 9$

The point A on H has coordinates $\left(6, \frac{3}{2}\right)$.

(a) Show that the normal to H at the point A has equation

$$2y - 8x + 45 = 0$$

(5)

The normal at A meets H again at the point B.

(b) Find the coordinates of B.

(4)

$xy = 9 \qquad y = \frac{9}{x} = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = \frac{-9}{x^2} \Big|_{x=6} \qquad \frac{dy}{dx} = \frac{-9}{36} = \frac{-1}{4}$

$\therefore M_t = \frac{-1}{4} \Rightarrow M_n = 4 \qquad \left(6, \frac{3}{2}\right)$

$y - y_1 = m(x - x_1) \Rightarrow y - \frac{3}{2} = 4(x-6)$

$\Rightarrow y - \frac{3}{2} = 4x - 24 \Rightarrow 2y - 3 = 8x - 48$

$\therefore 2y - 8x + 45 = 0$ #

$y = \frac{9}{x} \Rightarrow 2y = \frac{18}{x} \Rightarrow \frac{18}{x} - 8x + 45 = 0$

$(\times x) \quad 18 - 8x^2 + 45x = 0 \Rightarrow 8x^2 - 45x - 18 = 0$

$\Rightarrow (x-6)(8x+3) = 0$

$\underline{x = 6}$ known $\qquad \underline{x = \frac{-3}{8}} \qquad y = \frac{9}{x} = \frac{9}{\frac{-3}{8}} = \frac{72}{-3}$

$= -24$

$B\left(-\frac{3}{8}, -24\right)$

Leave blank

6. (i) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n(4n^2 - 1)$$

(6)

$n=1 \quad \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$ LHS

RHS $n=1 \quad \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5-1) & 5^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$ $\therefore$ LHS=RHS $\therefore$ true for $n=1$

assume true for $n=k$

$\therefore \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix}$ is true.

$n=k+1 \quad \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) - 5^k & 5^k \times 5 \end{pmatrix}$

$= -\frac{1}{4}(5^k-1) - \frac{4}{4}(5^k)$

$= -\frac{1}{4}(5^k - 1 + 4 \times 5^k)$

$= -\frac{1}{4}(5 \times 5^k - 1)$

$= -\frac{1}{4}(5^{k+1} - 1)$

$\begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^{k+1}-1) & 5^{k+1} \end{pmatrix}$ $\therefore$ true for $n=k+1$

$\therefore$ true for $n=1$, if true for $n=k$ then true for $n=k+1$

$\therefore$ by Mathematical induction true for all $n \in \mathbb{Z}^+$ #

Question 6 continued

ii) $n=1$ $\sum_{1}^{1}(2r-1)^2 = (2-1)^2 = 1^2 = 1$ LHS

$\frac{1}{3}n(4n^2-1) = \frac{1}{3}(1)(4\times1^2-1) = \frac{1}{3}(4-1) = 1$ RHS

$\therefore$ LHS = RHS $\therefore$ true for $n=1$.

$n=k$ $\sum_{1}^{k}(2r-1)^2 = \frac{1}{3}k(4k^2-1)$ assume true.

$n=k+1 = \sum_{1}^{k+1}(2r-1)^2 = \sum_{1}^{k}(2r-1)^2 + (2(k+1)-1)^2$

$= \frac{1}{3}k(4k^2-1) + (2k+1)^2$

$= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2$

$= (2k+1)\left[\frac{1}{3}k(2k-1) + (2k+1)\right]$

$= \frac{1}{3}(2k+1)\left[k(2k-1) + 3(2k+1)\right]$

$= \frac{1}{3}(2k+1)\left[2k^2 - k + 6k + 3\right]$

$= \frac{1}{3}(2k+1)(2k^2+5k+3]$

$= \frac{1}{3}(2k+1)(2k+3)(k+1)$

$= \frac{1}{3}(k+1)\left[(2k+1)(2k+3)\right]$

$= \frac{1}{3}(k+1)\left[4k^2+8k+3\right]$

$= \frac{1}{3}(k+1)\left[4(k^2+2k)+3\right]$

$= \frac{1}{3}(k+1)\left[4[(k+1)^2-1]+3\right]$

$= \frac{1}{3}(k+1)\left[4(k+1)^2-1\right]$ $\therefore$ true for $n=k+1$

$\therefore$ true for $n=k$, if true for $n=k$ then true for $n=k+1$

$\therefore$ by Mathematical induction true for all $n \in \mathbb{Z}^+$

P 4 4 8 2 9 R A 0 1 7 2 8

Turn over

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7. (i)

$$\mathbf{A} = \begin{pmatrix} 5k & 3k-1 \\ -3 & k+1 \end{pmatrix}, \text{ where } k \text{ is a real constant.}$$

Given that **A** is a singular matrix, find the possible values of k.

(4)

(ii)

$$\mathbf{B} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}$$

A triangle T is transformed onto a triangle T' by the transformation represented by the matrix **B**.

The vertices of triangle T' have coordinates $(0, 0)$, $(-20, 6)$ and $(10c, 6c)$, where c is a positive constant.

The area of triangle T' is 135 square units.

(a) Find the matrix $\mathbf{B}^{-1}$

(2)

(b) Find the coordinates of the vertices of the triangle T, in terms of c where necessary.

(3)

(c) Find the value of c.

(3)

i) Singular $\Rightarrow$ $5k(k+1) - (3k-1)(-3) = 0$

$5k^2 + 5k + 9k - 3 = 0$

$5k^2 + 14k - 3 = 0 \quad (5k-1)(k+3)$

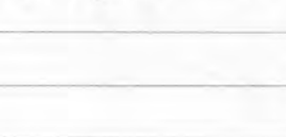
$\therefore k = \frac{1}{5} \quad k = -3$

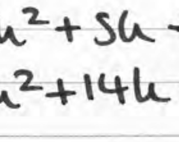
ii) B^{-1} $\quad \det B = 30 + 15 = 45$

$B^{-1} \ \frac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$

$T = \frac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}\begin{pmatrix} 0 & -20 & 10c \\ 0 & 6 & 6c \end{pmatrix} = \frac{1}{45}\begin{pmatrix} 0 & -90 & 0 \\ 0 & 0 & 90c \end{pmatrix}$

$T = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2c \end{pmatrix} \quad (0,0) \ (-2,0) \ (0,2c)$





Area $= 2c = \frac{135}{45} = 3$

$\therefore c = \frac{3}{2}$

8. The point $P(3p^2, 6p)$ lies on the parabola with equation $y^2 = 12x$ and the point S is the focus of this parabola.

(a) Prove that $SP = 3(1 + p^2)$ **(3)**

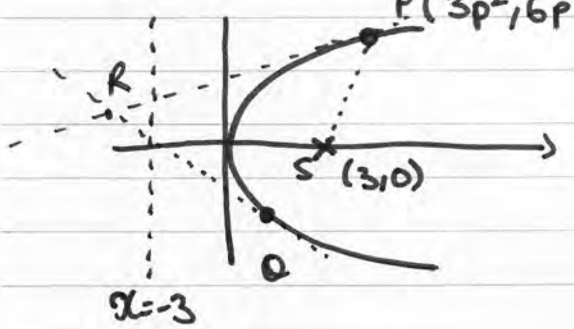
The point $Q(3q^2, 6q)$, $p \neq q$, also lies on this parabola.

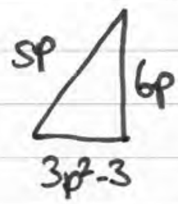
The tangent to the parabola at the point P and the tangent to the parabola at the point Q meet at the point R.

(b) Find the equations of these two tangents and hence find the coordinates of the point R, giving the coordinates in their simplest form. **(8)**

(c) Prove that $SR^2 = SP.SQ$ **(3)**

$y^2 = 4ax = 12x = \quad \therefore a = 3 \quad \therefore \text{focus } S(3,0)$





$SP^2 = 36p^2 + (3p^2-3)^2$

$= 36p^2 + 9p^4 - 18p^2 + 9$

$= 9p^4 + 18p^2 + 9$

$= 9(p^4 + 2p^2 + 1)$

$SP^2 = 9(p^2+1)^2$

$\therefore SP = 3(p^2+1)$ #

$y^2 = 12x$

$\frac{d}{dx}y^2 = \frac{d}{dx}(12x) \Rightarrow 2y\frac{dy}{dx} = 12 \quad \therefore \frac{dy}{dx} = \frac{6}{y}\Big|_{y=6p}$

$M_t \text{ at } P = \frac{1}{p}$

$y - y_1 = m(x - x_1) \Rightarrow y - 6p = \frac{1}{p}(x - 3p^2)$

$py - 6p^2 = x - 3p^2 \quad \therefore py = x + 3p^2$

$qy = x + 3q^2$

Q) $M_t \text{ at } Q = \frac{1}{q} \quad \therefore qy = x + 3q^2$

$py - qy = 3p^2 - 3q^2$

$y(\cancel{p-q}) = 3(p+q)(\cancel{p-q})$

$y = 3(p+q)$

$x = qy - 3q^2 = 3q(p+q) - 3q^2$

$x = 3pq + \cancel{3q^2} - \cancel{3q^2}$

$\therefore R(3pq, 3(p+q))$

P 4 4 8 2 9 R A 0 2 4 2 8

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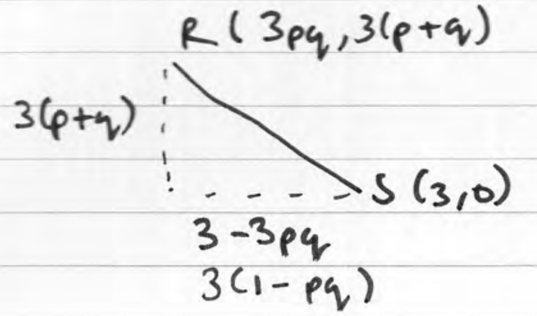
Question 8 continued

c) $SP = 3(p^2+1) \qquad SQ = 3(q^2+1)$



$$SR^2 = 3^2(p+q)^2 + 3^2(1-pq)^2$$

$$= 3^2\left[(p+q)^2 + (1-pq)^2\right]$$

$$= 3^2\left[p^2 + \cancel{2pq} + q^2 + 1 - \cancel{2pq} + p^2q^2\right]$$

$$= 3^2\left[p^2q^2 + p^2 + q^2 + 1\right]$$

$$= 3^2\left[(p^2+1)(q^2+1)\right]$$

$$= 3(p^2+1) \times 3(q^2+1)$$

$$\therefore \; SR^2 = SP \cdot SQ \quad \#$$